AD-A070 212 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
A PROOF OF THE CONVEXITY OF THE FREE BOUNDARY FOR POROUS FLOW T--ETC(U)
APR 79 C W CRYER DAAG29-75-C-0024
UNCLASSIFIED MRC-TSR-1953 NL

1 OF 1
AD
A070212

END
DATE
FILMED
7-79
DDC

MRC Technical Summary Report #1953

# A PROOF OF THE CONVEXITY OF THE FREE BOUNDARY FOR POROUS FLOW THROUGH A RECTANGULAR DAM USING THE MAXIMUM PRINCIPLE

C. W. Cryer

**Mathematics Research Center**
**University of Wisconsin–Madison**
**610 Walnut Street**
**Madison, Wisconsin 53706**



April 1979

(Received March 21, 1979)

See 1473 in back





Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

# A PROOF OF THE CONVEXITY OF THE FREE BOUNDARY FOR POROUS FLOW THROUGH A RECTANGULAR DAM USING THE MAXIMUM PRINCIPLE

C. W. Cryer*

Technical Summary Report #1953
April 1979

ABSTRACT

We consider the steady two-dimensional flow under gravity of water from one reservoir to another reservoir through a porous rectangular isotropic homogeneous dam with impervious bottom. Using the maximum principle we give a proof of the convexity of the free boundary.

AMS (MOS) Subject Classifications: 35J65, 76S05

Key Words: Porous flow, Rectangular dam, Maximum principle, Free boundary, Convexity

Work Unit Number 3 - Applications of Mathematics

*Computer Sciences Department and Mathematics Research Center, University of Wisconsin-Madison, Madison, WI 53706.

Sponsored by the National Science Foundation under Grant No. MCS77-26732. Support facilities provided by the United States Army under Contract No. DAAG29-75-C-0024.

## SIGNIFICANCE AND EXPLANATION

We consider the steady two-dimensional flow under gravity of water from one reservoir (on the left) to a lower reservoir (on the right) through a porous rectangular isotropic homogeneous dam with impervious bottom. Because of gravity the water does not flow through the entire dam and the dam is dry near its upper right corner. The interface separating the dry and wet regions of the dam is a free boundary. Recently, Friedman and Jenkins have proved that the free boundary is convex. We give a different proof which uses only the maximum principle and its generalizations.



---

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A PROOF OF THE CONVEXITY OF THE FREE BOUNDARY FOR POROUS FLOW THROUGH A RECTANGULAR DAM USING THE MAXIMUM PRINCIPLE

C. W. Cryer*

## 1. The Dam Problem

We consider the following Dam Problem. Water flows steadily under gravity through a porous rectangular isotropic homogeneous dam ABCF with impervious bottom BC of length L from a reservoir of height H to a reservoir of height h (see Figure 1.1). The water-air interface AE is a free boundary which we denote by $\Gamma$.

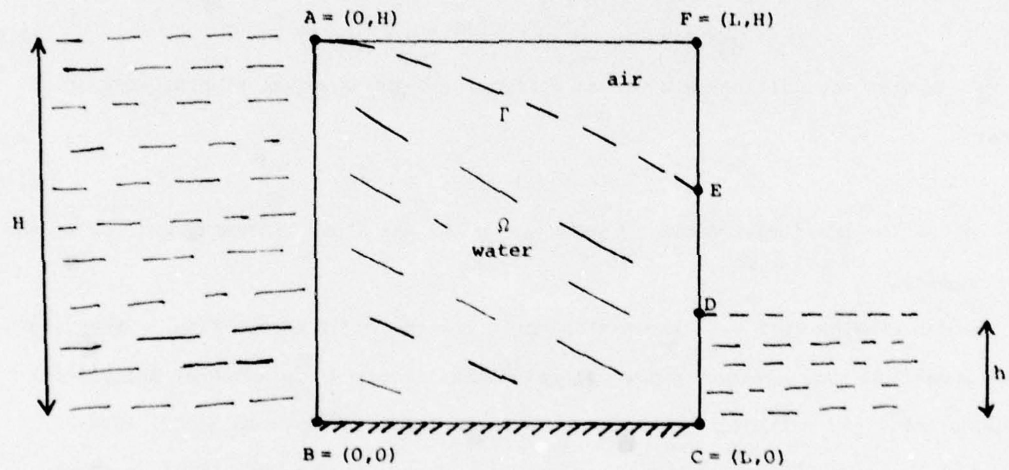


Figure 1.1: Flow through a rectangular dam.

The mathematical problem is as follows (Bear [1972], Baiocchi [1972], Baiocchi and Capelo [1978]):

Find functions $\varphi(x)$ (the height of the free boundary) and $u(x,y)$ (the hydraulic head) such that (from the equation of continuity and Darcy's law):

*Computer Sciences Department and Mathematics Research Center, University of Wisconsin-Madison, Madison, WI 53706.

Sponsored by the National Science Foundation under Grant No. MCS77-26732. Support facilities provided by the United States Army under Contract No. DAAG29-75-C-0024.

$$\mathrm{div}(\mathrm{grad}\ u) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = u_{xx} + u_{yy} = \nabla^2 u = 0, \quad \text{in } \Omega , \tag{1.1}$$

together with the boundary conditions,

$$u = H, \quad \text{on AB}, \quad \text{(interface with water at rest)} , \tag{1.2}$$

$$\frac{\partial u}{\partial n} = 0, \quad \text{on BC}, \quad \text{(impervious boundary)} , \tag{1.3}$$

$$u = h, \quad \text{on CD}, \quad \text{(interface with water at rest)} , \tag{1.4}$$

$$u = y, \quad \text{on DE}, \quad \text{(interface with air)} , \tag{1.5}$$

$$u = y, \quad \text{on EA}, \quad \text{(interface with air)} , \tag{1.6}$$

$$\frac{\partial u}{\partial n} = 0, \quad \text{on EA}, \quad \text{(streamline)} . \tag{1.7}$$

Here, $\Omega$ is the (unknown) domain,

$$\Omega = \{(x,y) : 0 < y < \varphi(x),\ 0 < x < L\} , \tag{1.8}$$

and $\frac{\partial}{\partial n}$ denotes the unit outward normal derivative. The physical significance of $u$ is that

$$u = y + p/\rho g , \tag{1.9}$$

where $g$ is the acceleration due to gravity, $p$ is the fluid pressure, and $\rho$ is the fluid density.

The dam problem is a well-known problem in the theory of porous flow. Cryer [1976, p. 54] summarizes the numerous numerical solutions. There are also three analytical solutions involving elliptic integrals: the first is due to Davison [1932, 1936, 1936a]; the second is due to Hamel [1934] and is described by Muskat [1937, p. 303] and Bear [1972, p. 398]; the third is due to Polubarinova-Kochina [1962, p. 284] (concerning misprints see Cryer [1976, p. 54]). Although these analytical solutions are rather complicated, they can, and have, been evaluated numerically.

Despite the fact that the analytical solution of the dam problem is known, the problem is still frequently considered in the literature because it serves as a useful model problem for porous flow free boundary problems. In a pioneering paper, Baiocchi [1972] reformulated the problem as a variational inequality for the function

$$w(x,y) = \int_y^{\varphi(x)} [u(x,t) - t]dt , \tag{1.10}$$

and derived many properties of $w$. Further properties of $w$ and $u = y - w_y$ have been obtained by Caffarelli [to appear], Friedman [1976], Jensen [1977], and Friedman and Jensen [1977]. Aitchison [1972] gives an expansion for the solution near the separation point $E$, and Aitchison [1977] gives numerical solutions.

Recently, Friedman and Jensen [1977] have proved that $\Gamma$ is convex under the following assumptions (which had previously been proved in the above-mentioned work of Friedman, Jensen, and Caffarelli):

$$u \in C(\bar{\Omega}) \cap C^2(\Omega) , \tag{1.11}$$

$$\varphi \in C[0,L] \cap C^2(0,L) . \tag{1.12}$$

$$\varphi \text{ is strictly monotone decreasing for } 0 < x < L , \tag{1.13}$$

$$u_y \text{ is bounded on } \Omega . \tag{1.14}$$

Making the same assumptions we give a different proof of convexity which uses only the maximum principle and its generalizations. Some preliminary results of general applicability are given in Section 2, and are then applied to the case in hand in Section 3.

The basic ideas in Section 3 have been known to us for almost twenty years and were originally motivated by proofs of convexity for fluid dynamics free boundary problems (Birkhoff and Zarantonello [1957, p. 84], Gilbarg [1960, p. 373]). Since the approach is effective in many fluid mechanics free boundary problems we hope that the same will be true for porous flow free boundary problems.

The present paper illustrates the power of the maximum principle as a tool for analyzing free boundary problems. It is appropriate to mention that the maximum principle was probably first applied to the dam problem by Davison [1936] and Shaw and Southwell [1941]. Some additional references to the application of the maximum principle to free boundary problems are given by Cryer [1977, Section II.13].

## 2. Preliminary Results on the Maximum Principle and Differential Geometry

We will require several versions of the maximum principle for a real function $u \in C(\Omega)$ which is harmonic in a bounded domain $\Omega$ with boundary $\partial\Omega$ in the complex $z = x + iy$ - plane. We assume that $u$ is continuous in $\bar{\Omega}$ except perhaps at a finite number of points $z_1, z_2, \ldots, z_m \in \partial\Omega$.

The Hopf Principle (Protter and Weinberger [1967, p. 65], Gilbarg and Trudinger [1977, p. 33]): Let $u(z) \leq M$ in $\Omega$, and $u(z_0) = M$ at a point $z_0 \in \partial\Omega$, the boundary of $\Omega$. Let (i) $u$ be continuous at $z_0$ and (ii) there exist a ball $B \subset \Omega$ with $z_0 \in \partial B$.

Then the outer normal derivative of $u$ at $z_0$, if it exists, satisfies

$$\frac{\partial u}{\partial n}(z_0) > 0 . \qquad \square$$

The Maximum Principle states that if $u \in C(\bar{\Omega})$ and $u \leq M$ on $\partial\Omega$ then $u \leq M$ in $\Omega$. In the present context, this principle is not adequate because we sometimes consider harmonic functions which are possibly not continuous at the boundary points $z_1 = A$, $z_2 = D$, and $z_3 = E$ (Figure 1.1). We will say that $u$ satisfies the Generalized Maximum Principle if

$$u(z) \leq M, \quad z \in \partial\Omega; \quad z \neq z_j, \qquad 1 \leq j \leq m , \tag{2.1}$$

implies

$$u(z) \leq M \quad \text{in} \quad \Omega .$$

Remark 2.1

If $v(\zeta)$ is harmonic in the unit circle $D$ and $v$ is continuous on $D$ except at $\zeta = 1$, it is not necessarily true that $v$ satisfies the Generalized Maximum Principle as is shown by the following examples.

$$v(\zeta) = v(re^{i\theta}) = v(r,\theta) = \text{Real}\left(\frac{1+\zeta}{1-\zeta}\right) = \frac{(1-r^2)}{1+r^2-2r\cos\theta} ,$$

is a function which is defined on the unit circle $D$ and $v = 0$ on $\partial D$ except at the point $\zeta = +1$, but $v(r,0) \to +\infty$ as $r \to 1$.

$$v(\zeta) = v(r,\theta) = \mathrm{Imag}\left[\frac{1+\zeta}{1-\zeta}\right]^2 = \frac{2r(1-r^2)\sin\theta}{[1+r^2-2r\cos\theta]^2},$$

satisfies $v(r,\theta) \to 0$ as $r \to 1$ for all $\theta$. □

In order for the Generalized Maximum Principle to apply, $u$ must satisfy some boundedness condition. In particular, it is known (Goluzin [1969, p. 267]) that if $u$ is bounded above in $\Omega$ then $u$ satisfies the Generalized Maximum Principle. The following lemma gives another condition.

Lemma 2.1

Let $v$ be the harmonic conjugate of $u$ in $\Omega$. ($v$ is well-defined and single-valued since, by assumption, $\Omega$ is a domain and hence simply connected.) Let $\partial\Omega$ be a rectifiable closed Jordan curve. If $|v| \leq M_v$ in $\Omega$ for some constant $M_v$ then $u$ satisfies the Generalized Maximum Principle.

Proof: Let $z = \omega(\zeta)$ denote the conformal mapping of the unit disk $D = \{\zeta : |\zeta| < 1\}$ in the $\zeta$-plane onto $\Omega$. The functions $u(z)$ and $v(z)$ are harmonic conjugates in $\Omega$, and so the functions $u(\omega(\zeta))$ and $v(\omega(\zeta))$ are harmonic conjugates in $D$.

We recall (Goluzin [1969, p. 385]) that the *Hardy-Lebesgue space* $h_2$ *consists of* the class of functions $\phi$ defined on $D$, which are harmonic on $D$, and which satisfy

$$\int_0^{2\pi} |\phi(r,\theta)|^2 d\theta < m$$

for some constant $m$ and all $r \in (0,1)$.

Since $v$ is bounded, $v(\omega(\zeta)) \in h_2$; hence, by the Riesz theorem, $u(\omega(\zeta)) \in h_2$ (Goluzin [1969, p. 392]).

Let $\phi(r,\theta) = \phi(\zeta) = u(\omega(\zeta))$. Since $\phi \in h_2$ the boundary values $\phi(1,\theta)$ are defined almost everywhere and Poisson's formula holds (Goluzin [1969, p. 391, Theorem 3], Rudin [1966, p. 232]); that is,

$$\phi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \phi(1,\theta)\,\frac{1-r^2}{1-2r\cos\theta+r^2}\,d\theta \ .$$

Let $u$ satisfy (2.1). Since $\partial\Omega$ is a rectifiable Jordan curve, the mapping $\omega$ associates sets of measure zero on $\partial\Omega$ with sets of measure zero on $\partial D$ (Goluzin [1969, p. 420, Theorem 2). Thus $|\phi(1,\theta)| \leq M$ a.e. on $\partial D$. Remembering that the weight function in Poisson's formula is non-negative, it follows from Poisson's formula above that $|\phi| \leq M$ on $D$. $\square$

Remark 2.2

If $u$ and $v$ are harmonic conjugates and $v$ is bounded then $u$ need not be bounded as shown by the example

$$u + iv = i \ln(x + iy) . \qquad \square$$

Further comments on the Generalized Maximum Principle will be found in the appendix.

In the dam problem we are given the values of $u$ and $u_n$ on $\Gamma$. The following lemma summarizes the relationships between the derivatives of a function on a curve. The lemma differs from the usual Frenet formulas (Eisenhart [1940, p. 25]) in that the curvature $\kappa$ may be either positive or negative.

Lemma 2.2

Let the boundary $\partial\Omega$ of $\Omega$ be defined parametrically by $x = \xi(s)$, $y = \eta(s)$, where $s$ denotes arc length along $\partial\Omega$ in the positive direction so that $\Omega$ is to the left of $\partial\Omega$ (Figure 2.1).

Let $\kappa = d\theta/ds = \dot{\theta}$ denote the signed curvature, so that if $\kappa$ is positive than $\Omega$ is convex. Let $\underline{t}$ be the unit tangent, and $\underline{n}$ the unit outward normal. Then

$$\underline{t} = (t_1, t_2) = (+\dot{\xi}, +\dot{\eta}) ,$$

$$\underline{n} = (n_1, n_2) = (+\dot{\eta}, -\dot{\xi}) ,$$

$$\dot{\xi}^2 + \dot{\eta}^2 = 1, \qquad \dot{\xi}\ddot{\xi} + \dot{\eta}\ddot{\eta} = 0 ,$$

$$\kappa = \dot{\xi}\ddot{\eta} - \ddot{\xi}\dot{\eta} ,$$

$$\ddot{\xi} = -\kappa\dot{\eta}, \qquad \ddot{\eta} = +\kappa\dot{\xi} ,$$

$$\frac{d\underline{t}}{ds} = -\kappa\underline{n}, \qquad \frac{d\underline{n}}{ds} = +\kappa\underline{t} .$$

Let $\psi = \psi(x,y)$ be twice continuously differentiable. We denote by $\psi_s$ the derivative of $\psi$ along $\partial\Omega$; that is,

$$\psi_s = \frac{d\psi(\xi(s), \eta(s))}{ds} .$$

We denote by $\psi_n$ and $\psi_t$ the directional derivatives of $\psi$ along $\underline{n}$ and $\underline{t}$; that is,

$$\varphi_n = n_1 \frac{\partial \varphi}{\partial x} + n_2 \frac{\partial \varphi}{\partial y}, \qquad \varphi_t = t_1 \frac{\partial \varphi}{\partial x} + t_2 \frac{\partial \varphi}{\partial y}.$$

Similarly,

$$\varphi_{nt} = n_1 t_1 \frac{\partial^2 \varphi}{\partial x^2} + (n_1 t_2 + n_2 t_1) \frac{\partial^2 \varphi}{\partial x \partial y} + n_2 t_2 \frac{\partial^2 \varphi}{\partial y^2},$$

$$\varphi_{ns} = \frac{d}{ds} (\varphi_n(\xi(s), \eta(s))) .$$

Then, on $\partial\Omega$,

$$\varphi_t = \varphi_s, \qquad \varphi_{tt} = \varphi_{ss} + \kappa \varphi_n,$$

$$\varphi_{nt} = \varphi_{ns} - \kappa \varphi_s . \qquad \square$$

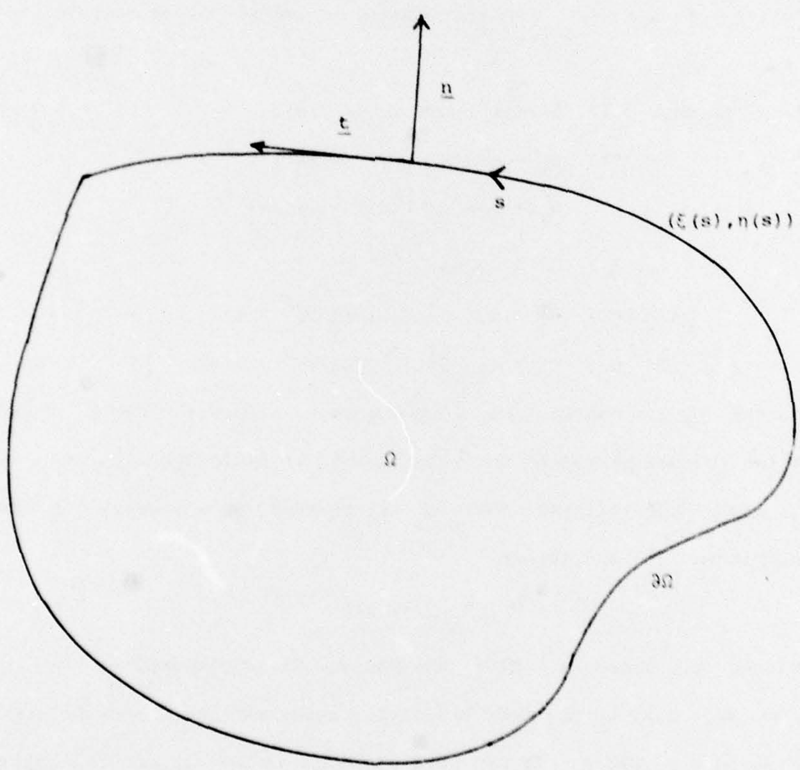


Figure 2.1: The curve $\partial\Omega$.

## 3. Convexity of the Free Boundary

The basic idea is to consider the function

$$U = u - y/2 \, , \tag{3.1}$$

where $u$ is the solution of the Dam Problem. We begin by obtaining estimates for $U$, $u$, and their derivatives, all of which are harmonic functions.

The solution $u$ can be reflected by symmetry across BC (Courant and Hilbert [1962, p. 272]) so that $u$ is infinitely differentiable near BC. It thus follows from the basic regularity theory for elliptic equations (Gilbarg and Trudinger [1977, Section 6.4]) that $u$ is twice continuously differentiable on $\bar{\Omega}$ except possibly at the points $z_1 = A$, $z_2 = D$, and $z_3 = E$. In this section, statements about the values of the derivatives of $u$ and $U$ on $\partial\Omega$ should be understood to exclude the points A, D, and E.

Using (1.2) through (1.7) direct computation yields:

$$\text{On EA:} \quad u_x = +\varphi'(x)/[1 + (\varphi'(x))^2] < 0 \, ,$$

$$u_y = +(\varphi'(x))^2/[1 + (\varphi'(x))^2] > 0 \, , \tag{3.2}$$

$$Q^2 \equiv |\text{grad } U|^2 = 1/4 \, .$$

$$\text{On ABCD:} \quad u_y = 0, \quad Q^2 = |\text{grad } U|^2 \geq 1/4 \, , \tag{3.3}$$

$$\text{On DE:} \quad u_y = 1, \quad Q^2 = |\text{grad } U|^2 \geq 1/4 \, . \tag{3.4}$$

The fact that $Q$ is constant on $\Gamma$ means that in porous flow $Q$ plays a role which is similar to that played by the velocity $q$ in fluid dynamics.

Since it was assumed ((1.14)) that $u_y$ is bounded, we may apply the Generalized Maximum Principle to $u_y$ and obtain

$$0 \leq u_y \leq 1, \quad \text{in } \Omega \, . \tag{3.5}$$

Now consider $u$. Since $u \in C(\bar{\Omega})$ the Maximum Principle implies that $u$ attains its extrema on $\partial\Omega$, but, by the Hopf Principle, these extrema cannot be attained on BC because $\partial u/\partial n = 0$ there. It has been assumed that $\Gamma$ is monotonically decreasing, so that $h \leq u \leq H$ on $\Gamma$. It follows that $h \leq u \leq H$ on CDEAB and so

$$h \leq u \leq H \quad \text{in } \Omega \, . \tag{3.6}$$

Next, we consider $u_x$. By the Hopf Principle applied to $u$ on AB and CD we see that

$$u_x < 0 \quad \text{on} \quad AB \cup CD \ , \tag{3.7}$$

where we have used the fact that $u$ can be reflected across BC. The boundary conditions for $u$ together with (3.6) imply that the harmonic function $u - y$ is non-negative on $\bar{\Omega}$. But $u - y = 0$ on DE and hence, by the Hopf Principle,

$$u_x < 0, \quad \text{on} \quad DE \ . \tag{3.8}$$

On BC we have $u_y = 0$; noting (3.5), the Hopf principle implies that $(u_y)_y \geq 0$. That is,

$$(u_x)_x = -(u_y)_y \leq 0, \quad \text{on} \quad BC \ .$$

But $u$ is smooth on BC, $u_x(B) < 0$, and $u_x(C) < 0$. Thus,

$$u_x < 0, \quad \text{on} \quad BC \ . \tag{3.9}$$

The functions $u_x$ and $-u_y$ are harmonic conjugates in $\Omega$ and $u_y$ is bounded. Applying Lemma 2.1 and noting (3.2) as well as (3.7) through (3.10), we conclude that

$$U_x = u_x \leq 0 \quad \text{in} \quad \Omega \ . \tag{3.10}$$

Thus, $u_x < 0$ in $\Omega$ since otherwise the strong maximum principle (Courant and Hilbert [1962, p. 326]) implies that $u_x \equiv 0$ in $\Omega$, which is not possible. Hence

$$|\text{grad } U|^2 \geq u_x^2 > 0 \quad \text{in} \quad \Omega \ . \tag{3.11}$$

Remark 3.1

$U$ decreases monotonically on ABC and increases monotonically on CDE. The fact that $|\text{grad } U|^2 > 0$ in $\Omega$ is also a consequence of a result of Walsh [1950, p. 318, last paragraph] on the critical points of harmonic functions. □

Since $U_x < 0$ in $\Omega$ the function

$$\ell n(-U_x + iU_y) = \frac{1}{2} \ell n[U_x^2 + U_y^2] + i \text{ arc tan}(-U_y/U_x) \ , \tag{3.12}$$

is regular in $\Omega$ and has bounded imaginary part in $\Omega$. From Lemma 2.1 and (3.2) through (3.4) we conclude that

$$Q^2 = U_x^2 + U_y^2 \geq 1/4, \quad \text{in} \quad \Omega \ . \tag{3.13}$$

Finally, since $\ell n\, Q$ is harmonic in $\Omega$ and $Q$ attains its minimum on $\Gamma$, the Hopf Principle implies that

$$\frac{\partial}{\partial n} \ell n\, Q = \frac{1}{Q} \frac{\partial Q}{\partial n} < 0, \quad \text{on } \Gamma \ . \tag{3.14}$$

We now use Lemma 2.2 to express $\partial Q/\partial n$ in terms of the boundary data for $u$ on $\Gamma$ and the curvature $\kappa$ of $\Gamma$. Direct computation shows that, on $\Gamma$,

$$U = \eta/2; \ U_t = U_s = \dot{\eta}/2 \ ;$$

$$U_n = u_n - \frac{1}{2} n_1 \frac{\partial y}{\partial x} - \frac{1}{2} n_2 \frac{\partial y}{\partial y} = + \dot{\xi}/2 \ ,$$

$$U_{tt} = U_{ss} + \kappa U_n = \ddot{\eta}/2 + \kappa\dot{\xi}/2 = \kappa\dot{\xi} \ ,$$

$$U_{nt} = U_{ns} - \kappa U_s = \ddot{\xi}/2 - \kappa\dot{\eta}/2 = -\kappa\dot{\eta} \ ,$$

$$U_{nn} = -U_{tt} = -\kappa\dot{\xi} \ .$$

Thus, since $Q^2 = U_n^2 + U_t^2$,

$$\begin{aligned} QQ_n &= U_n U_{nn} + U_t U_{nt} \\ &= -\kappa\dot{\xi}^2/2 - \kappa\dot{\eta}^2/2 \\ &= -\kappa/2 \ . \end{aligned} \tag{3.15}$$

Together, (3.14) and (3.15) show that $\kappa > 0$ so that $\Gamma$ is convex.

Appendix: Remarks on the Generalized Maximum Principle

It is natural to try to weaken the assumptions of Lemma 2.1 because this would simplify applications to free boundary problems. For example, in this paper the assumption that $U_y$ is bounded (see (1.14)) was only needed so that Lemma 2.1 could be applied. The proof that $U_y$ is bounded (Friedman [1976]) is not straightforward, and it would be nice if it could be avoided.

The requirement in Lemma 2.1 that $v$ be bounded can be replaced by the weaker condition that $v(\omega(\zeta)) \in h_2$, but this is not a very convenient hypothesis to check. Alternatively, one could require that $u + iv \in E_1(\Omega)$ (Goluzin [1969, p. 438], Priwalow [1956, p. 188]).

In applications we will usually know that

$$\iint_\Omega (u^2 + v^2)dxdy < \infty , \tag{A.1}$$

so that $u$ and $v$ both belong to the space $L_2H(\Omega)$ which consists (Hille [1962, p. 325]) of functions $\phi$ which are harmonic in $\Omega$ and satisfy

$$\iint_\Omega \phi^2 dxdy < \infty . \tag{A.2}$$

It is tempting to conjecture that if $u,v \in L_2H(\Omega)$ then $u,v \in h_2$, but this is not true as the following example shows. Let $\Omega = D$, so that $\zeta = z$. Set

$$u(z) + iv(z) = \sum_{n=1}^{\infty} z^n/\sqrt{n} . \tag{A.3}$$

Expanding $u$ and $v$ in Fourier series and using the orthogonality relations of $\sin n\theta$ and $\cos n\theta$ we see that

$$\int_0^{2\pi} u^2(r,\theta)d\theta = \int_0^{2\pi} v^2(r,\theta)d\theta = \sum_{n=1}^{\infty} r^{2n}/n . \tag{A.4}$$

Thus $u,v \in L_2H(D)$ but $u,v \notin h_2$. Furthermore, using a result given by Titchmarsh [1939, p. 163, Problem 15], we see that

$$f(z) = f(re^{i\theta}) = u + iv = \int_0^{\infty} \frac{t^{-1/2}}{e^t - z} dt , \tag{A.5}$$

so that $f(z)$ is continuous on $\bar{D}$ except at $z = 1$.

Gehring [1957] (see also Tsuji [1959, p. 186]) uses the maximal theorem of Hardy and Littlewood to prove that if $u \in L_2H(D)$ then, for almost all $\theta$, $u(z) = o([1 - |z|]^{-1/2})$ as $z \to e^{i\theta}$ in any fixed Stolz domain with vertex $e^{i\theta}$. Unfortunately, this result is not quite strong enough to prove that $u$ is bounded, even if we assume that $u$ is bounded on $\partial\Omega$ except at $z = 1$. Otherwise, by mapping D onto the right half plane we could conclude that a harmonic function on the right half plane which has bounded limits on the imaginary axis, and bounded growth near the real axis, must be bounded. This would constitute a substantial strengthening of the Phragmen-Lindelöf Principle (Protter and Weinberger [1967, p. 94]).

In conclusion, we observe that the behaviour of solutions of elliptic equations near corners has been considered by Oddson [1968], Kondrat'ev [1967], Miller [1967, 1971] and Grisvard [1969]. These results are not immediately applicable to the present problem because they require knowledge of the behaviour of the free boundary $\Gamma$ near its endpoints. Of course, once it is known that $\Gamma$ is differentiable at its endpoints then this work will yield useful information about the behaviour of the solution can be obtained.

## REFERENCES

References marked by an * are included for completeness but are not quoted in the text.


AITCHISON, J.: Numerical treatment of a singularity in a free boundary problem. Proc. Royal Soc. London A330, 573-580(1972).

AITCHISON, J.: The numerical solution of a minimization problem associated with a free surface flow. J. Inst. Maths Applics 20, 33-44(1977).

BAIOCCHI, C.: Su un problema di frontiera connesso a questioni di idraulica. Annali di Matematica pura ed applicata (IV), 107-127(1972).

BAIOCCHI, C. and CAPELO, A.: Disequazioni Variazionali e Quasivariazionali. Applicazioni a Problemi di Frontiera Libera, Vol.2. Problemi Quasivariazionali. Bologna: Pitagora Editrice, 1978.

BEAR, J.: Dynamics of Fluids in Porous Media. New York:American Elsevier, 1972.

BIRKHOFF, G. and ZARANTONELLO, E.H.: Jets, Wakes, and Cavities. New York: Academic Press, 1957.

CAFFARELLI , L.A.: The local regularity of a phreatic surface. To appear.

*COLLINGWOOD, E.F. and LOHWATER, A.J.: The Theory of Cluster Sets. Cambridge: Cambridge University Press, 1966.

COURANT, R. and HILBERT, D.: Methods of Mathematical Physics, Vol.2. New York: Interscience Publishers, 1962.

CRYER, C.W.: A survey of steady-state porous flow free boundary problems. Technical Summary Report No.1657, Mathematics Research Center, University of Wisconsin, Madison, 1976.

CRYER, C.W.: A bibliography of free boundary problems. Appendix 4: Mathematical Methods. Technical Summary Report No.1793, Mathematics Research Center, University of Wisconsin, Madison, 1977.

DAVISON, B. On the steady motion of ground-water through the earth dam.(Russian). Memoires de l'Institute Hydrologique 6(1932).

DAVISON, B.: On the steady two-dimensional motion of ground-water with a free surface. Philosophical Magazine (7)21, 881-903(1936).

*DAVISON, B.: On the steady motion of ground-water through a wide prismatic dam. Philosophical Magazine (7)21, 904-922(1936a).

*DAVISON, B. and ROSENHEAD, L.: Some cases of the steady two-dimensional percolation of water through ground. Proc. Royal Soc. London A175, 346-365(1940).

EISENHART, L.P.: An Introduction to Differential Geometry. Princeton: Princeton University Press, 1940.

FRIEDMAN, A.: A problem in hydraulics with non-monotone free boundary. Indiana University Math. J. 25, 577-592(1976).

FRIEDMAN A. and JENSEN, R.: Convexity of the free boundary in the Stefan problem and in the dam problem. Archive Rational Mech. Anal. 67, 1-24(1977).

GEHRING, F.W.: On the radial order of subharmonic functions. J. Math. Soc. Japan 9,(1957).

GILBARG, D.: Jets and cavities. In Handbuch der Physik Vol. IX/III, Fluid Dynamics, Flugge, S. (editor). Berlin: Springer, 1960.

GILBARG, D. and TRUDINGER, N.S.: Elliptic Partial Differential Equations of Second Order. Berlin: Springer, 1977.

GOLUZIN, G.M.: Geometric Theory of Functions of a Complex Variable. Providence, R.I.: American Mathematical Society, 1969.

GRISVARD, P.: Equations differentielles abstraites. Annales scientifiques de l'Ecole Normale Superiore (4)2, 311-395(1969).

HAMEL, G.: Uber Grundwasserstromung. Z. angew. Math. Mech. 14, 129-157(1934).

*HAYMAN, W.K. and KENNEDY, P.B.: Subharmonic Functions. London: Academic Press, 1976.

HILLE, E.: Analytic Function Theory, Vol. 2. Waltham, Mass.: Blaisdell, 1962.

*JENSEN, R.: Structure of the non-monotone free boundaries in a filtration problem. Indiana Univ. Math. J. 26, 1121-1135(1977).

*KINDERLEHRER, D. and NIRENBERG, L.: Regularity in free boundary problems. Annali Scuola Normale Superiore-Pisa, Classe di Scienze, (IV)4, 373-391(1977).

KONDRAT'EV, V.A.: Boundary problems for elliptic equations with conical or angular points. Trans. Moscow Math. Soc. 16, 227-313(1967).

*LEWY, H.: On the nature of the boundary separating two domains with different regimes. Atti del Convegno Internazionale "Metodi valutativi nella fisica-matematica", Accad. Naz. dei Lincei, 181-188(1975).

MILLER, K.: Barriers on cones for uniformly elliptic operators. Annali Mat. pura appl. 76, 93-105(1967).

MILLER, K.: Extremal barriers on cones with Phragmen-Lindelof theorems and other applications. Annali Mat. pura appl. 297-329(1971).

MUSKAT, M.: The Flow of Homogeneous Fluids through Porous Media. New York: McGraw-Hill, 1937.

ODDSON, J.K.: On the boundary point principle for elliptic equations in the plane. Bull. Amer. Math. Soc. 74, 666-670(1968).

POLUBARINOVA-KOCHINA, P. YA.: Theory of Groundwater Movement. Princeton: Princeton University Press, 1962.

PRIWALOW, I.I.: Randeigenschaften Analytischer Funktionen. Berlin: VEB Deutscher Verlag der Wissenschaften, 1956.

PROTTER, M.H. and WEINBERGER, H.F.: Maximum Principles in Differential Equations. Englewood Cliffs, N.J.: Prentice-Hall, 1967.

RUDIN, W.: Real and Complex Analysis. New York: McGraw-Hill, 1966.

SHAW, F.S. and SOUTHWELL, R.V.: Relaxation methods applied to engineering problems. VII. Problems relating to the percolation of fluids through porous materials. Proc. Royal Soc. London A178, 1-17(1941).

*SHIMBORSKY, E.: Encadrement d'une frontiere libre relative a un probleme d'hydraulique. Bollettino Unione Mat. Ital. (4)12, 66-67(1975).

TITCHMARSH, E.C.: The Theory of Functions. Oxford: Oxford University Press, 1939.

TSUJI, M.: Potential Theory in Modern Function Theory. Tokyo: Maruzen, 1959.

WALSH, J.L.: The Location of Critical Points. New York: American Math. Soc., 1950.

CWC/scr

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

| | | |
|---|---|---|
| 1. REPORT NUMBER 1953 | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle) A PROOF OF THE CONVEXITY OF THE FREE BOUNDARY FOR POROUS FLOW THROUGH A RECTANGULAR DAM USING THE MAXIMUM PRINCIPLE | | 5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s) C. W. Cryer | | 8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024 NSF-MCS77-26732 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Applications of Mathematics |
| 11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below. | | 12. REPORT DATE April 1979 |
| | | 13. NUMBER OF PAGES 15 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) MRC-TSR-1953 | | 15. SECURITY CLASS. (of this report) UNCLASSIFIED |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

Technical summary rept.

12 19p.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Porous flow
Rectangular dam
Maximum principle
Free boundary
Convexity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

~~We consider~~ The steady two-dimensional flow is considered under gravity of water from one reservoir to another reservoir through a porous rectangular isotropic homogeneous dam with impervious bottom. Using the maximum principle, ~~we give~~ a proof of the convexity of the free boundary is given.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

221 200